Entry Task: (directly from an old midterm)

You sell lollipops. Your profit, in dollars, from selling q thousand lollipops is given

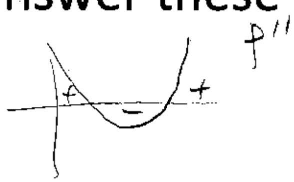
$$P(q) = q^4 - 32q^3 + 270q^2 - 200$$

(a) Find all the critical values.

(b) Find the min and max profit you can make if you sell between 1 and 10 thousand lollipops.

(c) Find the longest interval on which the profit function is concave down.

Without actually solving, quickly tell me the steps you would take to answer these questions.



(a) Step 1: FIND $P'(q)$

Step 2: SOLVE $P'(q) = 0$

$$4q^3 - 96q^2 + 540q \overset{?}{=} 0$$

$$\Rightarrow 4q(q^2 - 24q + 135) \overset{?}{=} 0$$

$$S\ \boxed{q=0} \text{ OR } q = \frac{24 \pm \sqrt{24^2 - 4(1)(135)}}{2} = \boxed{9 = q} \text{ OR } \boxed{15 = q}$$

(b) Step 1: Plug endpoints into profit.

Step 2: Plug appropriate critical value into profit

$$P(1) = (1)^4 - 32(1)^3 + 270(1)^2 - 200 = \boxed{39} \text{ MIN}$$

$$P(9) = \dots = \boxed{4903 \leftarrow \text{MAX}}$$

$$P(10) = \dots = 4800$$

(c) Step 1: FIND $P''(q)$

Step 2: SOLVE $P''(q) = 0$

$$P''(q) = 12q^2 - 192q + 540 \overset{?}{=} 0 \quad \div 12$$

$$q^2 - 16q + 45 = 0$$

$$q = \frac{16 \pm \sqrt{16^2 - 4(1)(45)}}{2} = 3.641 \text{ or } 12.359$$

P	∪		∩		∪
p''	+	3.641	−	12.359	+

11.1/2 Exponential and Logarithm Rule

Motivation - Recall from Math 111:

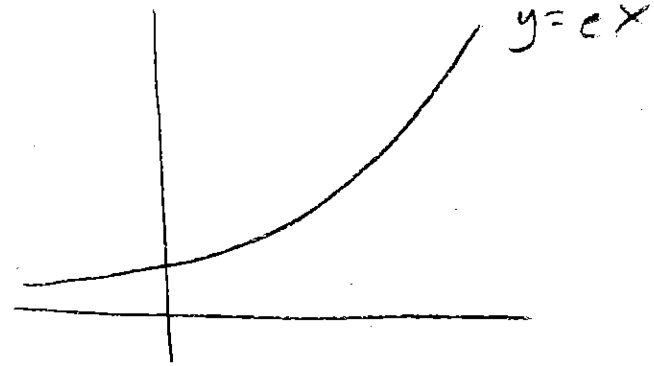
The functions $y = e^x$ and its inverse $y = \ln(x)$ are essential tools in finance.

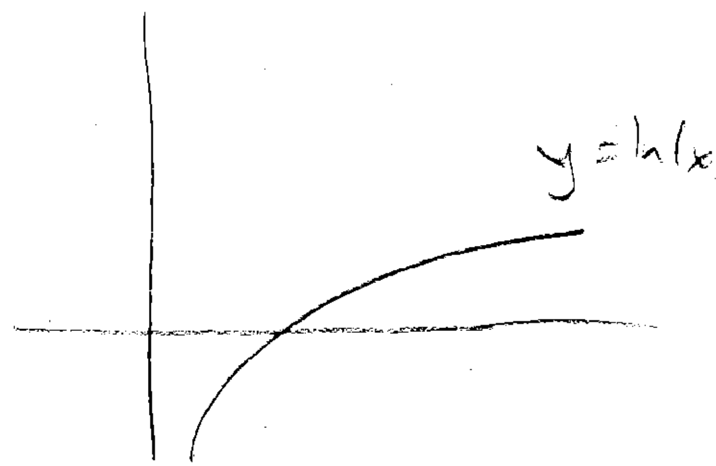
Such as:

Discrete Compounding: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Continuous Compounding: $A = P\,e^{rt}$

In both these formulas, you needed logarithms to solve for time.





This quarter, we have learned that derivatives are the key tools in analyzing any function. So if we are going to learn calculus for business, then we better also learn derivatives of $y = e^x$ and $y = \ln(x)$.

Basic Facts:

1. $y = e^x$ is the same as $\ln(y) = x$.

2. $\ln(e^x) = x$ and $e^{\ln(y)} = y$.

3. $1 = e^0$ and $\ln(1) = 0$.

4. $(e^a)^b = e^{ab}$ and $\ln(c^d) = d\ln(c)$.

5. $e^a e^b = e^{a+b}$ and

$$\ln(cd) = \ln(c) + \ln(d).$$

Solve $1000 = 200 e^{0.03t}$

$$5 = e^{0.03t}$$

$$\ln(5) = \underbrace{\ln(e^{0.03t})}$$

$$\ln(5) = 0.03t$$

$$t = \frac{\ln(5)}{0.03} \approx 53.6479$$

?

Solve

$$1000 = 200(1.03)^t$$

$$5 = (1.03)^t$$

$$\ln(5) = \ln(1.03^t)$$

$$\ln(5) = t\ln(1.03)$$

$$t = \frac{\ln(5)}{\ln(1.03)}$$

$$= 54.4487$$

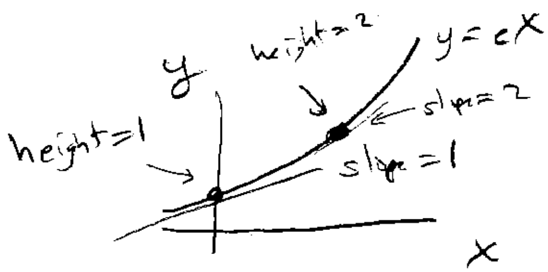
Rules so far - Sum, Coeff., Prod., Quot.,

$$\frac{d}{dx}(x^n) = n\,x^{n-1}.$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x),$$

which combine to make

$$\frac{d}{dx}\left((g(x))^n\right) = n(g(x))^{n-1}\cdot g'(x)$$



New rule

$$\frac{d}{dx}(e^x) = e^x$$

Combining with the chain rules gives

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\cdot g'(x)$$

$$\frac{f(x+h)-f(x)}{h} = \frac{e^{x+h}-e^x}{h} = \frac{e^x(e^h-1)}{h}$$

approaches 1

Examples: Differentiate

1. $y = e^{(x^3-5x^2)}$

$$\boxed{y' = e^{(x^3-5x^2)}\cdot(3x^2-10x)}$$

$e = 2.71828182\ldots$

$2. \ TC(q) = e^{\sqrt{q}}$

$= e^{(q^{1/2})}$

$$\boxed{TC'(q) = e^{(q^{1/2})} \cdot \tfrac{1}{2} q^{-1/2}}$$

$= \dfrac{e^{\sqrt{q}}}{2\sqrt{q}}$

$3. \ f(x) = e^{5/x^2} = e^{5x^{-2}}$

$$\boxed{f'(x) = e^{(5x^{-2})} \cdot (-10x^{-3})}$$

$= \dfrac{-10\, e^{5/x^2}}{x^3}$

$4. \ f(x) = (5x^3 + e^{7x})^{10}$

$$\boxed{f'(x) = 10\,(5x^3 + e^{7x})^9 \cdot (15x^2 + e^{7x} \cdot 7)}$$

Note: There is a big, big difference between a power function and an exponential function

Power function: $y = \left(f(x)\right)^n$

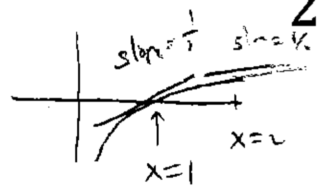
(variable only appears in the base)

Exponential function: $y = e^{f(x)}$

(variable only appears in the exponent)

New rule

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$



2. $f(t) = \ln(t^3 + e^t)$

$$f'(t) = \frac{1}{t^3+e^t} \cdot (3t^2+e^t)$$

$$= \frac{3t^2+e^t}{t^3+e^t}$$

Combining with the chain rules gives

$$\frac{d}{dx}(\ln(g(x))) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

Examples: Differentiate

1. $y = \ln(5x^4 - 3x^2)$

$$y' = \frac{1}{5x^4-3x^2} \cdot (20x^3-6x)$$

$$= \frac{20x^3-6x}{5x^4-3x^2}$$

3. $y = \ln\left(4\sqrt{t} + \frac{1}{t}\right) = \ln(4t^{1/2} + t^{-1})$

$$y' = \frac{1}{4t^{1/2}+t^{-1}} \cdot (2t^{-1/2} - t^{-2})$$

$$y' = \frac{2t^{-1/2} - t^{-2}}{4t^{1/2}+t^{-1}}$$

Finding Derivatives (SAME AS BEFORE!)

Step 0: Rewrite powers and simplify.

Step 1: Product, Quotient or Chain?

Chain could look like:

$(\text{BLAH})^n$, e^{BLAH}, or $\ln(\text{BLAH})$

Step 2: Use appropriate rule, in the middle of that rule you may need to do a derivative (back to step 1)

Examples: Differentiate

1. $y = \underbrace{\ln(2x+1)}_{F}\underbrace{e^{5x}}_{S}$

$$y' = \underbrace{\ln(2x+1)}_{F}\,\underbrace{e^{5x}\cdot 5}_{S'} + \underbrace{\frac{2}{2x+1}}_{F'}\,\underbrace{e^{5x}}_{S}$$

2. $g(x) = \dfrac{\sqrt{x}}{1+e^{x^4}} = \dfrac{x^{1/2}}{1+e^{x^4}}$ ← N ← D

$$g'(x) = \frac{(1+e^{x^4})\frac{1}{2}x^{-1/2} - x^{1/2}\cdot e^{(x^4)}4x^3}{(1+e^{x^4})^2}$$

2. $h(t) = (\ln(3t^4+1))^{50}$

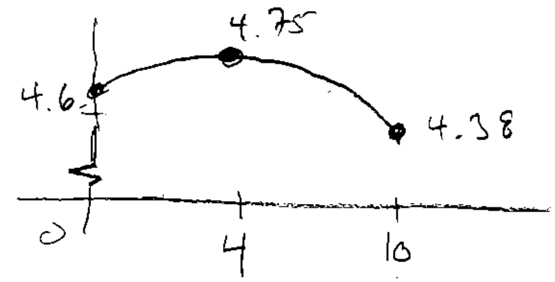
$$h'(x) = 50(\ln(3t^4+1))^{49}\cdot\frac{1}{3t^4+1}\cdot 12t^3$$

Quick Application

Find the global max and global min of

$$f(x) = \ln(100 + 8x - x^2)$$

on the interval x = 0 to x = 10.



$$f'(x) = \frac{8 - 2x}{100 + 8x - x^2} \stackrel{?}{=} 0$$

$$\Rightarrow \quad 8 - 2x = 0$$

$$\Rightarrow \quad 8 = 2x \Rightarrow x = 4$$ ← ONLY CRITICAL VALUE!

$$f(0) = \ln(100) \approx 4.6052$$

$$f(4) = \ln(100 + 32 - 16) = \ln(116) \approx 4.7536$$ ← GLOBAL MAX

$$f(10) = \ln(100 + 80 - 100) = \ln(80) \approx 4.3820$$ ← GLOBAL MIN